

AN INTRODUCTORY ANALYSIS OF THE THEORY OF PROBABILITY OPTION PRICING MODELS

FOR
THE OFFICE OF RESEARCH AT
MILAN STOCK EXCHANGE,
ITALY

BY

KLAUS SOLBERG SØILEN

DIRECTED BY

PROF. SALVATORI, BOCCONI

CONTENTS

PREFACE	3
INTRODUCTION.....	4
Options – An analogy.....	6
Volatility and price.....	7
The causes of volatility.....	7
Determining factors in the evaluation of an option.....	12
<i>Table 1: Determining factors and effects of increase</i>	12
GENERAL BUILDING BLOCKS FOR A COMPREHENSIVE OPTION MODEL.....	13
The Markov Analysis.....	13
The Monte Carlo process.....	14
Decision making under conditions of uncertainty.....	14
Decision making without probabilities.....	14
Decision making with probabilities.....	14
<i>Figure 1: Decision making tree with probabilities</i>	15
Bayesian analysis and decision analysis with additional information.....	16
THEORIES FROM PHYSICS USED IN OPTIONS THEORY.....	17
Thermodynamics from physics in options theory.....	17
<i>Figure 2: A Light Bulb</i>	18
<i>Figure 3: The equilibrium through the factor time</i>	18
From the physical problem to the corresponding mathematical model.....	19
<i>Table 2: From physical system to mathematical model</i>	19
The Lyapunow time horizon, or “the rock in the water”.....	20
THE DIFFERENT OPTION FORMULAS.....	21
The hedged portfolio argument.....	21
Sprenkle’s formula (1961).....	22
Samuelson’s model (1965).....	23
Samuelson and Merton’s model (1969).....	24
The Black-Scholes model (1972-73).....	25
Risk neutral valuation.....	26
Dividends.....	26
Deriving the Black-Scholes Formula.....	28
Diffusion processes.....	29
Geske’s Compound Option Approach.....	30
Cox, Ross, Rubinstein’s Binomial Approximation.....	32
Gatineau-Madansky Model.....	32
CONCLUSION.....	34
PRIMARY REFERENCES.....	35
SECONDARY REFERENCES.....	36

PREFACE

There is little that is fundamentally new in this paper, or that has not been published before either in books or scientific articles within the area of finance and capital markets. The paper is an attempt to present a series of models used to evaluate an option, to state the essential principles of them accurately, briefly to show the differences between them and to question the theory behind them.

The limits of the paper are many. The mathematics of the formulas and especially the Black-Scholes model have been recalculated by a fellow student at SDA Bocconi, Vladimir Baltaga, a student of physics from Moscow.

I would like to thank the office of research at the Milan stock exchange and Prof. Salvatori, at the Bocconi for their assistance and encouragement with this project. I would also like to thank the teachers in finance who inspired me, Prof. M. Marelli, Prof. M. Valetta, Prof. W. Lazzari at the SDA Bocconi, and Prof. B. Allaz at the HEC-ISA in France.

I would like to thank the administration at HEC/ISA for taking the risk of sending me on the exchange program to SDA Bocconi in Milan. I hope I have honored both institutions by my visit.

Milan, 1993

The Author

INTRODUCTION

Recent articles in books, scientific journals and the weekly press have showed a strong interest in the new techniques applied by analysts in the financial markets (The Economist, Oct. 9). The problems of finding the right price on a derivative is not first of all a question of following a specific procedure, but which procedure to follow and why. These questions lie at the heart of the problem in financial theory as the subject presents itself as a science, and outside most courses in business administration. In this paper we will focus on a small part of the theory of capital markets, namely options.

Economics, and more precisely theoretical economics, is the only one of the social sciences, which has inspired to the distinction of an exact science. To the extent that it is an exact science it must accept the limitations as well as share the dignity which belongs to that science, and in this way it becomes like physics or mathematics in being necessarily somewhat abstract and unreal. It is different from physics in degree, since though it cannot well be made so exact, yet for special reasons it secures a moderate degree of exactness only at the cost of much greater unreality. The very conception of an exact science involves abstractions, its ideal is analytical treatment, and analysis and abstraction are virtually synonyms.

It is a fact that the application of an analytical method in any class of problems, especially in what concern social systems, is always very incomplete. It is still not possible to deal with a very large proportion, numerically speaking, of the vast complexity of factors entering into normal real situations such as we find them in our practical everyday lives. So, simplifications are needed.

The value of a model depends on the fact that in large groups of problem situations certain elements are common and are not merely present in each single case, but in addition to that are both few in number and important enough largely to explain the situations. The law of these few elements enable us to reach an approximation to the law of the situation as a whole. Probability models give us statements of what "tends" to hold true or "would" hold true under "ideal" conditions, meaning merely in a situation where the numerous and variable less important "other thing" which our law do not take into account were entirely absent.

In physics, the model or archetype of an exact science of nature, a relatively small and workable number of laws or principles tell us what would happen if simplified conditions be assumed and all the disturbing factors were eliminated. The simplified conditions include specifications as to dimensions, mass, shape, smoothness, rigidity, elasticity and properties generally of the objects worked with, specifications usually quite impossible to realize in fact, yet absolutely necessary to make, while for "disturbing factors" are simply anything not included in the specifications, and their actual elimination is probably equally impossible to realize, and, again, equally necessary to assume. Only in this way can we obtain what we call a "law", descriptions of the separate elements og phenomena and their separate behavior. And while such laws, or models never accurately hold good in any particular case, because they are incomplete, not including all the elements in the case, yet they enable us to deal with practical problems intelligently because they are approximately true and we know how to discount for their incompleteness. We will discuss this further when we talk about the evaluation of an option.

There are quite different ideas in the financial market as how to best find the right price of an option. At the one extreme we have a tendency to repudiate abstraction and deduction all together, and insist only on purely objective, descriptive procedures. At the other extreme there is a strong and perhaps growing tendency of mathematical economists and pure theorists who consider anything outside of a closed model as a "gut feel". We shall look at the enjeu more carefully.

Options – An analogy

We shall start by describing what an option is through the use of an example. If you want to buy some apples the price you are likely to pay for them will depend on how much you want them relative to how much money you have. You might not want to eat them yourself, but sell them, in which case you will have to consider how much you can get for them, or how much you can get for the product of which the apples are a part, e.g. apple juice.

If you are thinking of making apple juice and selling it during local football matches in the fall, your decision of whether or not to buy the apples will depend on their price at the time when the apples are ripe to pick. If we are in the month of May the apples haven't started to grow yet and we do not know whether it is going to be a good harvest or not. These are elements of uncertainty which lies at the heart of most major problems in business. Because there is uncertainty there is risk. Our task is not to make the risk disappear, which we can not, as we do not decide the weather, but to handle the type of problem called risk as to reduce it to a minimum.

I have an apple garden. If you promise to pay a certain amount of money for some of my apples now before they have started growing, I will give you the right to come and pick them when they are ripe for a fixed amount of money so you can start to plan your sales in the fall, but only if you want to. Notice, that in the case you pay for the right to pick them and *have* to pick them whether you want them or not and pay the price we decided, we call it not an option, but a future, or in case of a larger non-standardized agreement, a warrant. Thus, an option gives you more flexibility and freedom of choice.

You are buying the choice of picking them, this because you have certain expectations as to the sales of apples, or rather apple juice, in the fall. You buy the choice because the resources, in this case, apples are scarce, this is one of the chief assumptions underlying every problem in economics. If there were apples to everybody and everybody could pick them and make apple juice from them there would be no market where the product could be sold. Instead people would pick apples for their own consumption only. Economics first start when some people realize that they may develop an advantage handling a specific resource and developing what we call a market, that is a meeting place where sellers and buyers can meet, making other people want to buy that product. A reason why many would want to buy that product is because they are better at presenting other things to the market. This is what we refer to as the situation of specialization, which has been the major drive to the development of new markets.

When buying the right to pick them, you risk e.g. that they rot. If that happens you lose the money you paid for the right to pick them. We realize that the components we pay for are the time until purchase, the expected value of the apples based on last years price and estimated preferences in the market. The longer the time until purchase, the larger the risk for both seller and buyer.

If I tell you you can have the right to pick the apples for an amount which sounds very cheap, you may start to wonder: Does he know something I do not know? That is, does he have information I do not have, which makes him more capable than I of saying what the supply and demand of apples is going to be at the time when the apples are ripe? In most cases the seller and the buyer will not tell why they think and handle as they do, that is they stick to

their information. If they convey their information and both parties act rationally on that information there will be no demand for apples. A market situation appears when the parties have different expectations of where the market is going. The bigger the convergences, the better the possibilities for trade in the market. What the one person win is what the other person loose.

The same goes for the primary market as for the secondary market, that is for apples as for options on the apples. An option market is nothing else than a place where people with different expectations about the future meet to make deals about the possibility of purchasing some scarce resource. They buy or sell, not the product, but the possibility of buying it. This is the essence of the option.

Volatility and price

Let us look further into the main question for the buyer and the seller, namely the price. What will it cost you to buy an option of the apples to be picked when they are ripe? Will the price be the same as last year's? In this case you would base your expectation on historical price, which is the same as to say that the future will be like the past. Even though we know that that is partially the case, e.g. when predicting the weather, it will often be wrong and when it is wrong you loose money.

Our ability to predict the price will depend on our ability to select a number of relevant variables and make correct predictions as to theses variables. Our formulas are nothing but an expression of this. One variables may be the volatility of the price. This allows us to set up a linear regression which can serve as the basis for a formula. The formulas used in the calculations of an option happen to be well known in physics.

How important is it that you know the volatility of the price in order to guess the correct price of the options on the apples? The answer is, very important. As e.g. Fabozzi and Modigliano (1992) show in their book on capital markets the Black-Scholes formula (BS), which is the most well known, will only give a correct value under limited volatility. We can check this ourselves with the help of a pocket calculator and a probability table. What this means is that if your expectations are very different from the historical price, then you risk making a wrong decision as to the price to be paid for the option.

You might say that this has made you more uncertain as to what price to pay for the option. You are right, if you do not have sufficiently strong expectations as to the volatility of the option's price, you may want to take the risk of waiting until the apples are ripe. It all depends on the price of the option and on the transactions costs. Maybe they are so low it really doesn't matter if the apples turn out to be rotten.

The causes of volatility

Proponents of the efficient markets hypothesis have traditionally claimed that the volatility of a stock is caused solely by the random arrival of new information about the future returns from the stock, as all information is reflected in the current price. Others have claimed that volatility is caused largely by trading. An interesting question, therefore, is whether volatility is the same when the exchange is open as when it is closed.

Fama and K. French have tested this question empirically. They collected data on the stock price at the close of each trading day over a long period of time, and then calculated:

1. The variance of stock price returns between the close of trading on one day and the close of trading on the next trading day when there are no intervening non trading days.
2. The variance of the stock price return between the close of trading on Fridays and the close of trading on Mondays.

If trading and non trading days are equivalent, the variance in situation 2 should be three times as great as the variance in situation 1. Fama found that it was only 22% higher. French's results were similar. He found that it was 19 percent higher.

These results seem to suggest that the volatility is far larger when the exchange is open than when it is closed. Proponents of the traditional view that volatility is caused only by new information might be tempted to argue that most new information on stock arrive during trading hours. The only reasonable conclusion seems to be that volatility is to some extent caused by trading itself.

What are so the implications of this on our model? If daily data are used to measure volatility, the results suggest that days when the exchange is closed can be ignored. The volatility per annum is calculated from the volatility per trading day using the formula

$$\begin{aligned} & \text{Volatility per annum} \\ & = \\ & \text{Volatility per trading day} \\ & \quad \times \\ & \text{The square root of the number of trading days per annum.} \end{aligned}$$

The precision with which volatility is measured increases as more information is used to estimate it. With actively traded assets, the precision of the estimate of volatility should be arbitrarily high given the large amount of trade price information. However, volatility changes. Volatility may indeed be considered stable over the remaining life of an option, consistent with standard option pricing formulas. However, events of the recent and not so recent past may provide a poor index of the market's anticipation of future volatility. As we use data extending further back in time, the estimate of volatility may become more precise. At the same time, it may prove less relevant.

EVALUATION OF AN OPTION

What goes into the price of an option? Is it the sum of all costs plus a profit? What kind of costs do we have? These are some of the first questions we will be asking. In corporate finance we learn that risk can be divided into systematic and unsystematic risk. That is, we can divide risk into company specific and industry specific risk, and even into country specific risk. The study of finance tells us that we can diversify away everything that is not company specific.

The next question: What is profit? We learn that it can be described mechanically, as a function of supply and demand. Without touching on the ethical question of profit, we can say that the market accepts this answer as we accept the law of gravity. In the old days they talked of profit as a price for having experienced stress. This stress was later substituted with the idea of risk, e.g. by, which is later made quantifiable and thereby to some extent predictable.

Price and profit has always been a primary question for economists. J.B. Say describes profit as a wage for the capitalist which included risk. Thünen calls it extra productivity of managers labor. Clark characterizes profit as the lure that causes men to make the efforts and take the risks involved in progress. At the end they all say more or less the same. Since Adam Smith price has been the resolution of a tension between supply and demand. We have all seen the graph showing the relationship between the two and their equilibrium. However, this procedure of finding the correct price has a difficult prerequisite, namely complete or perfect information which is hardly ever the case. In reality we never have perfect information. Instead, the equilibrium of supply and demand will always be a function of the information available at the moment of the decision.

In ancient times the "natural price" was the cost of production plus whatever was needed to survive. It was early pointed out however that the price not only should reflect that which was being sold, but also the loss which the sale could bring on the seller for not spending his or her money otherwise¹. In the theory of finance we talk of opportunity cost, one of the most fundamental concepts in economics according to the MIT dictionary of Modern Economics. Because of opportunity cost it is considered lawful to sell for more than it is "worth", putting pressure on the borrower to make even higher returns. The development of the question of the right price in economics pass by "just" to "natural" until "fair" in the literature which has come to mean any price the seller can get for his item with little consideration as to the way the price was obtained. So is it also with options.

This does not mean that prices do not have a history. The wheat was sold for a certain price yesterday. If today I make a bid which is much lower than that, I will probably get no wheat. If my price is much higher, I shall place myself at a disadvantage to my competitors, who may have bought at a lower price. This logic gives reason to what we call stochastic process, which is an essential assumption in any formula used to calculate the price of an option. A price which does not follow this process, but "jump" below or above yesterday's price, is referred to as a "jump" or a "poisson" process.

A pricing decision may come from a formula, a computer, or it may be a gut feeling, or even the result of tossing a coin (random walk), but it is a decision when we act upon it. In a

¹ See St. Thomas Aquinas, "Of cheating, which is committed in buying and selling", in Summa Theologica.

formula or a computer program there are already built-in decisions, or assumptions, which the price depends upon. It is these assumptions we want to examine closer.

Supply and demand – the fundamental elements of “pure” economics – are not acts of God or events of nature, but acts of human beings who define themselves by what they do in the market. If the market actors are not acting out of free will, economic discourses are tempted by the methods of the natural sciences and ideally physics. Brookway (1991) argues with Bacon’s system of the academic disciplines that economics and the study of law are both divisions of ethics. Today, he argues, motive and intention is central in law, but insignificant in economics.

Every human action can be studied from the point of view of all the arts and most of the sciences. Nothing I do is without physical aspects. There is never a time when we do not obey the laws of motion, when we can step off a moving train without getting hurt. And nothing we do is without an emotional reaction. We risk making other people angry if we treat them poorly. The complexity of the number of variables needed to explain a bullish market goes way beyond simple psychology. If you try to isolate the pure economic motives for an action you risk having to make so many assumptions that the result is of no substantial value for the decision maker. On the other hand if you do not make any assumptions you stand in front of a problem which you will have the difficulty of resolving.

One natural temptation has been to use mathematics in economics, simply because it involves numbers which are easy to handle. In the modern world of social sciences mathematics has rivaled the authority of psychology. Statistics, or as it was called first, political arithmetic, has experienced more success than any other field of mathematics in this respect for no other reasons than because that it allows for broad answers through probability calculations (Kelley, 1962). Statistics is also used in the derivation of our formulas, which build upon a Gaussian, or normal distribution curve.

The attraction of mathematics for economists is enhanced by the hope that somewhere and somehow they will discover something like Newton’s inverse square law. At first glance Newton’s problem does seem similar to that of the econometrician’s. Both are confronted by a universe of infinite detail and variety, and both are sustained by the hope that, somewhere in that booming, buzzing confusion, orderly or reliable laws may be discovered.

To sum up we shall say that the price is a function of supply and demand, that these factors change over time. If all variables were independent we could search out the important ones and solve a series of semilinear equations, in much the same way as the field of econometrics is developed². Unfortunately, life is not that simple; the variables are not independent. A change in inflation is likely to affect the interest rate. In order to solve a real life problem mathematically you would have to set up a non-linear, non homogenous differential equation with non constant coefficients. Even if we could write down the equation, we would not be able to solve it in closed form.

If I arrive at an amount x after having done the calculation for the value of an option, am I willing to give any price below this amount for the option? Why is the B-S model being used when it is based on assumptions about the market environment that are obviously false? The distinction between price and value reflects the not often recognized difference between theories of option evaluation and option pricing. The B-S model aims to derive the value of an

² The background foundation of macro econometrics had their origin in the nineteenth-century investigation dealing with various aspects of quantitative economics (Bodkin, Klein, Marwah, 1991)

option so that it consists with the price of the underlying stock. But, in trying to apply a theoretical valuation model to the real world, it is immediately clear that none of the model assumptions actually holds 100%. The arbitrage strategy that is risk less and cost less in theory, is neither in practice. There is risk because the position can't be re balanced continuously when markets are closed, and less-than-continuously re balancing can still lead to large transaction costs. Even the theoretical option value itself is uncertain, since it depends on the volatility of the stock that can be known exactly. Unlimited arbitrage does not dominate the market.

In actual markets, options prices, like prices for everything else, are determined by supply and demand. This includes the supply and demand for non-arbitrageurs. Few investors use the model mainly for finding misplaced options that can be arbitrated against the underlying stock. The major reason for that is that none feels confident that they know the true volatility. Options traders normally pay more attention to the implied volatility than the historical volatility.

The questions we have raised about option valuation models relate to option pricing in the real world have no easy answer. Figlewski (1989) suggests that what we really need is a model that explains how the market prices options. We should examine broader classes of theories that include factors like expectations, risk-aversion, and the market imperfection. It is safer to use the existing models for hedging than for computing option values.

Determining factors in the evaluation of an option

Any book on options will tell us that there are six major factors to consider in most of the option pricing formulas. These are:

1. The price of the underlying financial claim
2. The predetermined exercise price
3. The amount of time until the option expires
4. The interest rate on risk-free assets
5. The underlying volatility of the price optioned financial claim measured by the standard deviation of expected return, σ
6. The dividend on interest received on the optioned financial claim during the period the option is outstanding

All of this information is readily available in the market for those who know where to look for it, except for expected volatility, which we must look closer at. This is how each of the six factors influence the price of a call option:

1. The higher the price of the underlying, the higher the price of the call. The higher the market price, given a specific exercise price, the higher chance that the option will be exercised
2. The higher the exercise price, the lower the value of the option, or to use financial jargon, the less likely it is that the option will be in-the-money
3. The longer the option period, the higher the value of the option, Thus, the higher chance that the option will end up in-the-money
4. The higher the price volatility, the higher the value of the option. Or to say it with statistical jargon, the higher the standard deviation, the greater value of the option
5. The higher the interest rate, r , related to the cost of financing the underlying claim, the higher the option price³
6. The more dividends or interest that is being paid on a stock, the less the value of the option. This is tricky especially for longterm options who's underlying turns out unexpected payments

We can list the following determinants of the option value:

Table 1: Determining factors and effects of increase

Determining Factors		Effects of Increase	
		Put	Call
1	Current stock price (s)	Decrease	Increase
2	Striking price (k)	Increase	Decrease
3	Time to expiration (t)	Increase	Increase
4	Interest rates	Decrease	Increase
5	Stock volatility	Increase	Increase
6	Cash dividends	Increase	Decrease

³ Using an option allows the investor to control an asset without tying up funds through owing it. Therefore, the higher the interest rate, such as the cost of a margin loan to finance the asset, the more valuable the option.

GENERAL BUILDING BLOCKS FOR A COMPREHENSIVE OPTION MODEL

In this part we shall look at factors to consider when building a comprehensive model for the options market. The first step in this process is to try to identify a number of general building blocks (Dothan, 1990):

1. Time
 - a. The time period of the model
 - b. The data on which traders trade and consume
2. Uncertainty
 - a. A listing of all the basic events or states that could occur during the time period of the model and their probabilities
3. Exchange goods
 - a. The goods which traders receive, exchange, and consume, including money
4. Securities
 - a. contracts for a future delivery of exchange goods, contingent on the prevailing state
5. Traders
 - a. Also called consumers, investors, or agents, and the information available to them on trading and consumption dates
6. Endowments
 - a. Endowments of consumption goods that the traders receive from the source other than trading
7. Consumption possibilities
 - a. Consumption possibilities of traders during the time period of the model
8. Preferences
 - a. Preferences of the traders over their consumption possibilities during the time period of the model

When these items are specified, the model determines the demands of securities by traders, their consumption, and the price of securities on trading dates.

The simplest model is the One-Period-Model, with finite time interval and finite number of states. The next kind of model is the Discrete Multiperiod Model, which also has a finite number of states, but which receives and trades a finite number of states inside a certain time period. We may say that it has infinite time intervals inside the time period. The third kind is the Continuous Multiperiod Model. It has a finite number of states, and both the arrival of information and trading in securities are continuous. The first two models can be solved using linear algebra and some elementary probability statistics. For the last model will have to use continuous stochastic and the Markov process.

The Markov Analysis

The Markov analysis is not an optimization technique, but a descriptive technique that results in probabilistic information. The properties of the analysis are

1. The transition probabilities for a given beginning state of the system sum to 1.0
2. The probabilities apply to all participants in the system

3. The transaction probabilities are constant over time
4. The states are independent over time

A Markov process can be divided into Continuous time markov processes which are of two kinds: Poisson processes and Diffusion processes. Markov processes implies that all information is in the current price. It is a random function of two variables, X and t , where x is independent over time t .

The Monte Carlo process

The Monte Carlo process is another probability model used in the theory of options. The process is a way to select truly random numbers, like the ones you would find in a random number table, or in an "ideal" casino.

The technique can be defined as a technique for selecting numbers randomly from a probability distribution for the use in a trial run of a simulation. The basic principles behind the process is the same as in the operation of a gambling casino, with such devices as the roulette, dice or playing cards.

Decision making under conditions of uncertainty

In our example the demand for a product – an option – may be 100 units, or 50 or 200 at a given time, depending upon the state of the market (which is uncertain). Given this type of decision several decision making techniques are available to help the decision maker, belonging to two broad groups or classes: those where probabilities cannot be assigned to future occurrences, and those situations where probabilities can be assigned.

Decision making without probabilities

A decisions making situation includes several components. The decisions themselves and the actual events that will occur in the future, known as the states of nature. At the time the decision is made, the decision maker is uncertain as to which state of nature will occur in the future, and thus, he has no control over them.

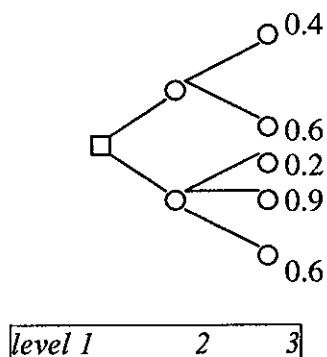
One techniques often used is the pay-off table. In general, a pay-off table is a mean of organizing and illustrating the pay offs from the different decisions given the various states of nature in a decision process.

Decision making with probabilities

In our example we did not have any information regarding the probable occurrence of the states of nature. In other words, no probabilities of occurrences were assigned to the states of nature. Introducing a real option we may find it useful to use a decision tree. A decision tree,

like the one we find in the Binomial Approach discussed later, consists of nodes and branches. Many will know this way of organizing data in time from a course in game theory. Rather than determining the probability of each branch as in a probability tree, in the decision tree expected outcome is computed and a decision can be made on the expected values. The primary gain of the decision making tree is nothing more than to illustrate the decision making process. This makes it easier to see what the possible decisions are, and it becomes easier to compute the expected values and understand the process of making decisions.

Figure 1: Decision making tree with probabilities



The square is the necessary starting point. The circles are decision nodes, reflecting the different possibilities from the previous decision if made. The probability added up in the end reflects the value of that decision where 1.0 is equal to absolute certainty. The deeper the level, the more actions have been made, and the less certain that the probability is correct. E.g. we can imagine that an option at level 1 has one probability for the price at n USD, when the same option at level two includes an option where the dividend is paid at price n_1+n_2 USD.

When a decision situation requires only a single decision, an expected value pay off table will yield the same result as a decision tree. A pay off is usually limited to a single decision situation. If a decision requires a series of decisions we need to use a decision tree in our analysis. A decision tree with several points, or squares, is called a sequential decision tree, but functions in exactly the same way.

Bayesian analysis and decision analysis with additional information

The concept of conditional probability given statistical dependence forms the necessary foundations for the area of probability known as Bayesian analysis. It is hardly surprising to the reader that we can make better decisions if we obtain perfect information. We can use a process for additional information in our decision tree applying Bayesian analysis. The basic principle of this analysis is that additional information (if available) can sometimes enable individuals to alter (improve) the marginal probabilities of the occurrences of an event. The altered probabilities are referred to as revised or posterior probabilities.

In general, given two events A and B, and a third C that is conditionally depended upon A and B, Baye's rule can be stated as:

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A) + P(C|B)P(B)}$$

THEORIES FROM PHYSICS USED IN OPTIONS THEORY

Whenever a physical law involves a rate of change of a function, such as velocity, acceleration, it will lead to a differential equation. Differential equations are of fundamental importance in engineering mathematics because many physical laws and relations appear mathematically in the form of such equations.

These two terms, velocity and acceleration, are enough to characterize most market variables for mathematical purposes. If all variables were independent one could search out the important ones and solve them by a series of semilinear equations, arriving ultimately at a dollar value for each set of conditions.

Much of the field of econometrics has evolved around this approach. Unfortunately, life is not that simple. A change in the rate of inflation will very likely affect the interest rate. There is a cross-coupling between most of the variables. Moreover, many interactions are non-linear. If we were to try to write the equation mathematically like the problem appears before with non constant coefficients. Even though we can write the equation down, we can't solve it in closed form⁴.

If we can not write the equation down, nor solve it even if we could write it down, what can we do with it? This question lies at the core of many real world problems in science. Following this, the analyst is forced to add indirect methods to understand what is happening in the market.

Thermodynamics from physics in options theory

Conceptually we can think of an equation in option as the function for thermodynamics in physics. A characteristic state variable is the price (P), while the composite net of supply and demand factors will be called the tendency (T). The equation describing this situation is:

$$P = f(T)$$

Or in other words, the price is a function of the tendency. As demand become stronger T increases and the price rises. To illustrate the example in physics, we will look at what happens inside of a light bulb. The equation is:

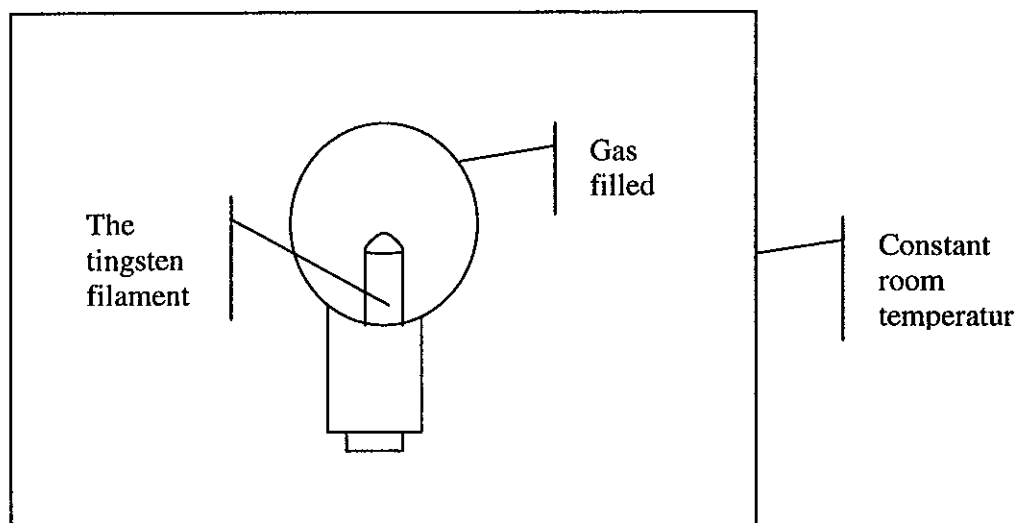
$$P = contT$$

Where P equals pressure and T is the temperature. When we turn on the light switch, the tungsten filament begins to heat up, warming the gas in the bulb. When T increases, the pressure P begins to rise. Very shortly the heat added is exactly balanced by the radiate losses from the bulb and the cooling of the bulb by the air. At this point, the temperature stops

⁴ By closed form we mean that you can get a new answer by changing one or several of the variables in the equation.

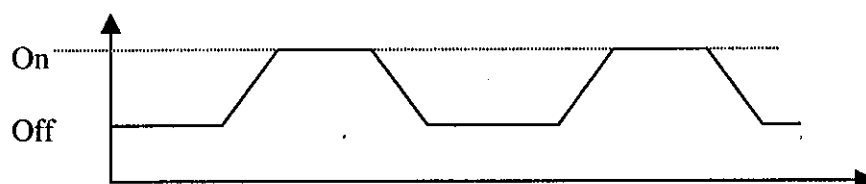
changing and the pressure now stays constant. The beginning state (light off) and the ending state (light on at constant T) are called equilibrium points. To go from one equilibrium to the next we must pass through a transition stage due to the change in temperature (T). When the light is switched off, the temperature falls and the pressure drops until equilibrium is again achieved. By looking at this process we might have an idea of how very idealized markets act.

Figure 2: A Light Bulb



The problem is to, so to speak, know when the light is switched on and off. How do we set up an equation which takes into consideration the different decisions of a board of directors, just to take an example, who themselves take decisions based on what happens in their micro and macro environment? Some believe it is impossible to quantify and understand all of the fundamental factors that go into the making of a run, or equilibrium period. However, the existence of an underlying equation of state, with state variables such as price, provide us with a framework for our analysis which can not be ignored. Most analysts will agree that the price behavior is far from random.

Figure 3: The equilibrium through the factor time



It is impossible to quantify and understand *all* of the fundamental factors that go into making of an equilibrium. However, the existence of an underlying equation of state, with state variables such as price, provides us with a framework for an analysis.

Once we have established that the supply-demand equation is fairly well balanced and regular we can use that information to explore, or project the future behavior from the past. This is the momentum concept. In physics it's the first law of motion (bodies in motion tend to remain in motion). In the economics affecting most stocks and their options, the underlying forces change slowly because of the basic inertia of the system.

From the physical problem to the corresponding mathematical model

The transition from the physical problem to the corresponding "mathematical problem" is called modelling, and is what has given the term "engineering" to the capital markets. Modelling means setting up mathematical models of physical or other systems, following the steps:

Table 2: From physical system to mathematical model

Modeling	Solving	Interpreting
Physical system	Mathematical model	

The golden rule in the procedure is that if you can not solve a given problem, try to solve a simpler one first, that is, try to simplify and see what is possible, and then work your way up to more advanced models. You should characterize the classes of equations which are possible⁵. Hence, we often start by looking for an integrating factor depending only on one variable.

A stochastic variable, or a random variable x , is a function associated with an experiment whose values are real numbers and their occurrences in the trials depends on "chance".

A model can be created to explain or forecast price change. Most models are only good for explanation, not for forecasting. That is, they tell you why things went as they did, not how they are going to happen. Explanatory models analyze sets of data at concurrent times, they look at relationships between multiple factors that can effect on the price at the same moment in time. They can also look for casual, or "lagged" relationships, where prices respond to other factors after one or more days. It is possible to use the explanatory model to determine the normal price at a particular moment. Although not considered forecasting, any variation in the actual market price from the normal or expected price could present trading opportunities.

Methods of selecting the best forecasting model can effect its credibility. An analytic approach selects the factors and specifies the relationships in advance. Tests are then performed on the data to verify the premises. Many models, though, are refined by "fitting" the data, using regression analysis, which applies a broad selection of variables and weighting to find the best fit. These models do not necessarily forecast but are definitely optimized hindsight. On the other side, an analytical approach that is subsequently "fine tuned" could be in danger of losing its forecasting qualities.

⁵ In the same way the Norwegian mathematician Nils Henrik Abel (1802-1829) showed that you can not solve an equation by the 5th degree.

The Lyapunow time horizon, or “the rock in the water”.

This is an important theory in options. We assume that people will act in the same way to a similar happening. That is, we assume that given no dramatic change the future will be like the past. When we have an expected event (the rock touching the surface of the water), the computer tries to find a similar pattern of trade from the past in the data bank.

THE DIFFERENT OPTION FORMULAS

The problem of finding an explicit option valuation formula was first addressed in detail in 1900 by the french mathematician Bachelier. He advanced a long way along the path that would develop into modern option pricing theory seventy years later. The major stumbling blocks encountered by financial reserachers trying to obtain an explicit option valuation model was that it seemed inherently to depend on two factors that could not be observed directly:

1. The probability distribution for the price of the underlying asset at option expiration
2. The option's expected payoff back to the present time value

It was in 1972 that Black and Myron Scholes developed a model which did not depend explicitly on the expected price movement of the underlying asset. This is referred to as the General Approach (1). The second large group of approaches derive from the Binomial model (2). In this paper we are basically going to treat the first and most well known group of models.

The hedged portfolio argument

The concept behind the Black-Scholes model is easily grasped, but the mathematics involved in deriving it are rather daunting. Over the years several "tricks" have been found that can greatly simplify the problem. One is the principle of "risk neutral" valuation. By placing the problem in the context of a risk less portfolio the fair value of an option can be found without knowledge of investors risk aversion. The option will be priced in the same way (given the price of its underlying asset) regardless of how risk averse they are. In particular, a call will have the same value as if investors were risk neutral, e.g. if they were indifferent to risk and cared only about return. It turns out to be fairly easy to derive security valuation formulas for a risk neutral market environment, and the principle of risk neutral valuation says that these formulas also hold for options in a general risk averse market.

We will now give a presentation of the more well known probability option models, in chronological order, where the focus is limited to their content and main differences.

Sprengle's formula (1961)

Most of the previous work on options before the B-S formula had been expressed in terms of warrants. However, these formulas were not complete, since they all involved one or more arbitrary parameters. The formula of Sprengle is written:

$$KxN(b_1) - k^*cN(b_2)$$

$$b_1 = \frac{\ln kx/c - \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}}$$

$$b_2 = \frac{\ln kx/c - \frac{1}{2}v^2(t^* - t)}{v\sqrt{(t^* - t)}}$$

where, x=stock price
 c=the exercise price
 t*=the maturity date
 t=the current date
 v²=the variance rate of return on the stock
 ln=the natural logarithm
 N(b)=the cumulative normal density function

However, k and K* are unknown parameters, where

K=the ratio of the expected value of the stock at the time the warrant matures to the current stock price
K*=the discount factor that depends on the risk of the stock

This first model describes fairly well the relationship between the probability distribution of the stock price changes and option or warrant value. It approximates the description of the probability approach already given and soon to be developed further.

Samuelson's model (1965)

Samuelson's model is a derivation of the model of Sprenkle's. His two unknown parameters are labelled α and β , where

α = the rate of expected return on the stock, and

β = the rate of expected return on the warrant or the discount rate to be applied to the warrant.

He assumes that the distribution of possible values of the stock when the warrant matures is log-normal and takes the expected value of this distribution, cutting it off at the exercise price. He then discounts this expected value to the present at the rate β . Unfortunately there seems to be no model for the pricing under conditions of capital market equilibrium that would make this an appropriate procedure in determining the value of the warrant.

Samuelson and Merton's model (1969)

In a subsequent paper Samuelson and Merton recognizes the fact that discounting the expected value of the distribution of possible values of the warrant when it is exercised is not an appropriate procedure.

They advance the theory by treating the option price as a function of the stock price. They also recognise that the discount rates are determined in part by the requirement that investors are willing to hold all of the outstanding amount of both the stock and the option. But they do not make use of the fact that investors must hold other assets as well, so that the risk of an option or stock that affects its discount rate is only that part of the risk that cannot be diversified away. Their final formula depends on the shape of the utility function that they assume for the typical investor, which gives:

$$V_c = e^{r(t-t^*)} \int_{S/P_s}^{\infty} (ZP_s - S) dQ[Z:(t^* - t)]$$

Where,

V_c = fair value option

r = interest rate

t = current date

t^* = maturity date of option or warrant

\int_{S/P_s}^{∞} = integral over integral from $\frac{S}{P_s}$ to ∞

S = striking or exercise price

P_s = stock price

Z = random variable return per dollar invested in common stock

$dQ[Z:(t^* - t)]$ = risk - adjusted probability density function of Z over a time period of length $t^* - t$

e = base of natural log arithmes = 2.71828

The unique feature of this model is that it is based on what the authors call "utilprobe", that is, a combined utility and probability distribution.

The Black-Scholes model (1972-73)

Most students of financial economics know that the work of Black and Scholes (1973) on option pricing represents one of the major theories in modern finance. Of course, that is not without good reason. We shall therefore use more time to look at this model.

The model, which is derived by a differential equation, has two important contributions, of which the first one is the most obvious. The model helps us to find a fair price of an option (A). The other contribution is the assumption that it is possible to set up a perfectly hedged position consisting of a long position in an underlying stock and a short position in options on that stock, or vice versa, thereby completely eliminating market risk from the stock portfolio (B). This last discovery, it is often argued, is the real contribution of the formula to the field of capital markets.

The ideas behind the model are as follows: Suppose there is a formula that tells how the value of a call option depends on the price of the underlying stock (1), the volatility of the stock (2), the exercise price (3), the maturity of the option (4), and the interest rate (5).

The formula will tell us among other things how much the option value will change when the stock price changes by a small amount within a short time. Under the assumption that we can make a riskless position, mentioned above, we get that if the stock goes up, then we lose on the option but make it up on the gain on the stock. If the stock goes down, we will lose on the stock but make it up on the option.

The assumptions underlying the model are as follows:

1. There are two assumptions about risk free interest rate, r . The first is that the interest rate for borrowing and lending is the same for all maturities (1), and the second is that it is constant and known over the life of the option (2)
2. The model assumes that the stock price follows a stochastic (random walk) process called a diffusion process⁶, which simply means that if the price goes from say 100 to 120 then it has to take on all the values in between.
2b. Thus the distribution of possible stock prices at the end of any finite interval is log normal.
2c. Thus the variance rate of return on the stock is constant.
3. The stock pays no dividend and makes no other distributions. In the case of a dividend-paying stock it may be advantageous for the holder of the call option to exercise the option early.
4. The option is a European option, meaning the option can only be exercised at maturity.
5. The model ignores taxes and transaction costs. All securities are perfectly divisible.
6. It is possible to borrow any fraction of the price of a security to buy it or hold it, at the short-term interest rate.
7. A seller who does not own a security (a short seller) will simply accept the price of the security from the buyer and will agree to settle with the buyer on some future date paying him an amount equal to the price of the security on that date. While this short sale is outstanding, the short seller will have the use of, or interest on, the proceeds of the sale.

⁶ As opposed to the jump process of Merton, Cox and Ross which we will look at later.

8. Security trading is continuous.

Let us look at some of these assumptions in detail:

Risk neutral valuation

The formula's differential equation does not involve any variables that are affected by the risk preferences of investors. The variables that do not appear in the equation are the current stock price, time, stock price volatility, and the risk free rate of interest. All are independent of risk preferences.

The equation would not be independent of risk preferences if it involved the expected return on the stock, μ . This is because the value of μ does depend on risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given stock. It is fortunate that μ happens to drop out in the derivation of the equation.

The fact that the BS differential equation is independent of risk preferences enable an ingenious argument to be used. If risk preferences do not enter into the equation, they cannot affect its solution. Any set of risk preferences can therefore be used when evaluating f . In particular the very simple assumption that all investors are risk neutral can be made.

In a world where investors are risk neutral, the expected return on all securities do not require a premium to induce them to take risk. It is also true that the present value of any cash-flow in a risk neutral world can be obtained by discounting its expected value at the risk free rate. The assumption that the world is risk neutral therefore considerably simplify the analysis of derivative securities. Consider a derivative security such as a European option that pays off some function of the stock price at time T . First, the expected return from the stock is r rather than μ . This expected value is then discounted to the present time using a discount rate of r .

It is important to realize that the risk neutrality assumption is merely an artificial device for obtaining solutions to the BS differential equation. The solutions that are obtained are valid in all worlds – not just those where investors are risk neutral. When we move from a risk neutral world to a risk averse world, two things happen:

1. The expected growth rate in the stock price changes, and
2. The discount rate that must be used for any payoffs from the derivative security changes.

It happens that these two effects always offset each other exactly.

Dividends

The Black-Scholes model assumes that the stock upon which the option is written pays no dividends. In practice, this is not usually true. In this section we will assume that the dividends that will be paid during the life of an option can be predicted with certainty.

A dividend-paying stock can reasonably be expected to follow the stochastic process except when the stock goes ex-dividend. At this point the stock's price goes down by an amount reflecting the dividend paid per share. For tax reasons, the stock goes down by somewhat less than the cash amount of dividend. To take account of this, the word "dividend" in this section should be interpreted as the reduction in the stock price on the ex-dividend date caused by the dividend. Thus, if a dividend of \$1 per share is anticipated and the share price normally goes down by 80% of the dividend on the ex-dividend date, the dividend should be assumed to be \$0.80 for the purpose of the analysis.

European options can be analysed by assuming that the stock price is the sum of two components:

1. A riskless component that will be used to pay the known dividends during the life of the option, and
2. A risky component.

The riskless component at any given time is the present value of all the dividends during the life of the option discounted from the ex-dividend dates to the present at the risk-free rate. The dividends will cause the riskless component to disappear by the time the option matures. The Black-Scholes formula is therefore correct if S is put equal to the risky component. Operationally, this means that the Black-Scholes formula can be used provided that the stock price is reduced by the present value of all the dividends during the life of the option., the discounting being done from the ex-dividend dates at the risk-free rate.

One shouldn't be surprised to see the B-S formula written in different ways in different textbooks. It's all the same: we'll pick the easiest one to write down:

$$C = SN(x) - Kr^{-t}N(x - \sigma\sqrt{t})$$

$$\text{where, } x = \frac{\log(S/Kr^{-t})}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

The first line ($C=...$) we can explain fairly easily. The second part ($x=...$) is more difficult to explain, and would require more mathematics than the author has had. The price of the call is equal to the price of the stock minus the present value of the striking price. We'll go through each step of the formula:

1. Stock price: $\partial C / \partial S = N(x) \geq 0$
2. Striking price: $\partial C / \partial K = -r^{-t}N(x - \sigma\sqrt{t}) \leq 0$
3. Time to expiration: $\partial C / \partial t = (S\sigma^2/2\sqrt{t})N(x) + Kr^{-t}(\log r)N(x - \sigma\sqrt{t}) \geq 0$
4. Volatility: $\partial C / \partial \sigma = S\sqrt{t}N(x) \geq 0$
5. Interest rate: $\partial C / \partial r = tKr^{-(t+1)}N(x - \sigma\sqrt{t}) \geq 0$

The first point gives us the Delta. The third gives us the Theta. We'll look at two other measures, the gamma and the Omega:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{1}{S\sigma\sqrt{t}}$$
$$\Omega = \frac{S \Delta}{C} = \frac{SN(x)}{C}$$

The delta tells us that if the delta is .4, then the price of the call goes up with .4, when the stock price goes up with 1. The Omega gives us the answer in percentages: It says that when the omega is e.g. 1.4, the price of the call goes up with 1.4% when the price of the stock goes up with 1%.

Deriving the Black-Scholes Formula

Black used the capital asset pricing model to write down how the discount rate for a warrant varies with time and the stock price. This gave him the differential equation. According to the author he didn't recognize the equation as a version of the "heat equation" in physics at that time, and set the work on the formula aside, until he started to cooperate with Myron Scholes in 1969.

The real breakthrough in the work came when the two authors realized that they could diversify away all the risk. If the beta of the stock were zero, the beta of the option would have to be zero too. If the option always had an expected return equal to the interest rate, then the discount rate that would lead the option's expected future value to its value would always be the interest rate. The discount rate would not depend on time or on the stock price, as it would if the stock had an expected return other than the interest rate.

They then discounted the expected terminal value of the option at the constant interest rate to get the present value of the option, and took Sprenkle's formula, put in the interest rate for the expected return on the stock, and put in the interest rate again for the discount rate for the option. The formula was then checked against the differential equation. Voila!

Robert Merton also made some useful suggestions on the paper. In particular he pointed out that if you assume continuous trading in the option or the stock, you can maintain a hedged position between them that is literally riskless. In the final version of the paper, they derived it that way, because it seemed to be the most general derivation.

We'll go through the formula:

Let C be the theoretical value of a European call option. The objective is to derive an equation that expresses C as a function of the strike price X , time to expiration T , and the characteristics that have been assumed for the market environment and for the behavior of the underlying asset.

The first step is to look at a hedged portfolio containing one call option and some position in the underlying asset in the correct proportions so that whatever the price change on the asset is over the next instant, the profit and the loss on the two components of the portfolio offset each other.

It is useful to compare this to the way in which the Binomial model was developed. In that case, we used the underlying asset and riskless borrowing to create a portfolio that replicated the payoff pattern of a call option. Here we return to the relationship among these securities around, combining positions in the asset and the call option into a riskless investment. The basic arbitrage relation at work is the same.

Let ∂C represent the amount that the value of the call option will change if the price of the underlying asset move by a tiny amount ∂S . The ratio $\partial C/\partial S$ is the hedge ratio or the delta of the call for this set of parameters. We will use the symbol δ for delta.

Now think about the portfolio that is long one call option and short δ units of the underlying asset. If the asset price S moves by some amount y , the call value will change by δy . The value of the short position in δ units of the underlying asset will change by $-\delta y$, i.e., the same amount in the opposite direction. The position is therefore riskless overall for the next instant.

In the binomial model, the amount invested in the risky and the risk free assets were recalculated at every step. The same thing occurs in continuous-time models. Once the price moves, the delta changes too, and the position must be re balanced by buying or selling the some amount of the asset in order to achieve a new delta. That is why the price process must be continuous, to allow the continuous re balancing that keeps the hedge portfolio riskless at every instant in time.

Under the assumptions above; it is possible to write the stochastic differential equation for the movement in the value of the call and the hedged portfolio. And since this portfolio is instantaneously riskless, to prevent the possibility of arbitrage profits over the next interval dt it must return the riskless rate of interest, $r dt$. This relationship produces a "second order partial differential equation that must be satisfied by the call value formula.

Diffusion processes

Diffusion processes belong to the general class of Markov processes. A Markov process is a random function on two (kinds of) variables, X and t , where X represents the "state" and t the "index". For our purpose, t refers to time and is a nonnegative real number, while $X(t)$ is the state of the system under consideration at time t and may well be a vector of variables. It could be the exchange rate between two currencies, the price of a particular stock, a vector of return on different securities in an investment portfolio, etc.

A defining characteristics of a Markov process is that the random increments to X are independent over time. In other words, the probability of going from state X_0 at time t_0 to another state X_1 at a later time t_1 is independent of what state the system has been in any time before t_0 . When $X(t)$ refers to the market price in a financial market, the Markov property implies ("weak form") market efficiency. Once the current price is known, there is no further information about future price movements that can be gained from consideration of prices in earlier periods.

The Binomial process of asset prices is a good example of a Markov process operating in discrete time. That is, the time variable t takes on only integer values and the state of the system is only defined at those points. By contrast, a diffusion process is continuous in time, with randomness impinging upon the system constantly.

Diffusions are one of two fundamental classes of continuous time Markov processes, the other being Poisson processes. In a diffusion, small changes in state occur continually. During a short interval of time only local movements can take place, but there is some changes no matter how infinitesimal the time interval. The prototype physical analogue to a diffusion is the Brownian motion of a tiny particle suspended in a fluid. Under constant bombardment by molecules of the fluid, its position continually changes by small random amounts. A graph of a path of such a particle would show that it followed a continuous, though erratic, trajectory.

Poisson or "jump" processes, on the other hand, are characterized by discontinuous changes of state (jumps) that occur only at random intervals. At a given instant there is either no change or a jump. Such a process could be used to describe the way the prime rate of interest moves over time, or any similar system of administered prices, such as fixed exchange rates. The Markov property is reflected in the fact that the probability of a jump occurring within the next time interval of a given is independent of how long it has been since the previous jump. While some work has been done on option valuation for Poisson processes, primary interest has focused on the diffusion model. A major reason for this is that jumps in the price of the underlying asset make riskless arbitrage procedure for deriving an option model does not work.

Let us now see how a diffusion process arises as the limited case of a discrete time stochastic process when the time interval between observations goes to zero. Consider a discrete time model of a variable X that has in each period an expected change μ and a random increment $\xi(t)$ that has a variance σ^2 . We can write the equation X as

$$X(t+1) = X(t) + \mu + \sigma \xi(t+1)$$

$$\text{with } E[\xi] = 0$$

$$\text{Var}[\xi] = 1$$

$$\xi(t) \text{ and } \xi(s) \text{ independent for } t \neq s$$

$$\text{or, } X(t+1) - X(t) = \Delta X(t) = \mu + \sigma \xi(t+1)$$

The random increment ξ may be drawn from any probability distribution with a finite variance. The binomial would be one possibility.

The constants μ and σ are known as the instantaneous mean and the standard deviation. More general processes in which the instantaneous mean and variance are allowed to depend on the current state and time are known as It

Geske's Compound Option Approach

Geske's interest in options applies to corporate finance. He sees options as a way to purchase the entire firm, and his argument is that a call option is really an option on the firm's assets. A call option on common stock is therefore best analyzed as an option on an option.

The model:

Geske's model improves on some of the empirical weaknesses of the B-S model, and it explains some of the tendency of stock price volatility to increase as the stock price declines. It has two more variables than the B-S model.

One of the problem with this and other more complicated models is that they deal with too many unknowns, even though this makes more sense intuitively. You need too use several different probability distributions (nonlog-normal distribution) to determine the observed pattern of stock price changes. The model also uses data on the firm's leverage to determine the volatility of the stock.

Cox, Ross, Rubinstein's Binomial Approximation

Most of the alternative derivations of the B-S model have been of value primarily in explaining the mechanism behind the model from different perspectives. The use of the binomial approximation to the log normal distribution permits a more accurate adjustment for dividends. The approach also offers a possibility in dealing with the early exercise problem in put evaluation. Although a binomial model requires more computer memory and operating time than the traditional B-S model, the improvements in accuracy may be worth the cost.

Gastineau-Madansky Model

The model which is presented in Gastineau's "Stock Option Manual" from 1986 is very similar to the B-S model. The model, which is a probability model, can be adjusted for dividends, interest rates, option commission charges, and even tax rates. Instead of log normal distribution Gastineau uses the more complex empirical probability distribution.

$$V_C = a_1 e^{a_2 r(t-t^*)} \int_{a_3 S/P_S}^{\infty} a_4 (Z P_S - S) dQ[Z : (t^* - t) : a_5]$$

Where,

V_C = fair value of option

$a_1 \dots a_5$ = adjustment factors designed to reflect commission charges, dividends, interest rates, taxes, and other variables.

r = basic interest rate

t = current date

t^* = maturity date of option or warrant

$\int_{a_3 S/P_S}^{\infty} a_4$ = integral over interval from $\frac{a_3 S}{P_S}$ to ∞

∞ = infinity

S = striking price

P_S = Stock price

Z = random variable return per dollar invested in common stock

$dQ[Z : (t^* - t) : a_5]$ = an empirical probability density function of Z over a time period of length $t - t^*$

$e = 2.71828$

Each adjustment factor in this formulation may incorporate part or all of the adjustment for more than one variable. Just by looking at all these variables one immediately understands the

complexity of this formula, and thereby it's difficulty in use. It is interesting to notice that it has the concept of "fair price" built into it.⁷

⁷Gastineau dedicates a whole page in his option manual to the concept of "fair price".

CONCLUSION

The increased attention being devoted to the formal models of market equilibrium with taxes, does suggest that the Modigliani-Miller perfect-world models have become assimilated. Like their counterparts in physics, they have served to define the boundary limits within which acceptable solutions are constrained to lie. Finance theory is the branch of economics which has made most progress on optimal behavior in dynamic models with uncertainty.

It is worth pointing out that the crucial question for a buyer or a seller of an option is the derivatives value and when it should be exercised. And, techniques are of course meaningless without good data.

Mathematicians, statisticians, financial theorists and economists have found several interconnections in their research. The common theme so far is dynamic models in which uncertainty unravels over time.

Most research papers assume that agents (decision makers) behave optimally, that is, that they at each point of time use all information available to make the optimal decision concerning their options. This is not always the case; e.g. traders sometimes fail to exercise in-the-money positions.

Although many economists will agree that flexibility in the use of a formula has a value, it remains difficult to quantify. The remedy is to apply the methods of stochastic dynamic programming, or stochastic control. Optimal behavior is described by a strategy, which defines the decision variable (s) at each point in time as a function of the history up to, and including, that point.

The application of theories of optimal behavior becomes problematic the more unknown parameters there is in the model. The traditional way to meet this problem is first to try to reduce the number of parameters, and second, to find out if some parameters can be observed directly as prices or price ratios in the market. At the end the formulas are tested against empirical data.

If you trade at market price, you get the benefit of what the market knows. But it is not a good idea to insist on trading at the values given by the formula. The market may have a good reason to trade at another price.

PRIMARY REFERENCES

- Black, Fisher, and Scholes, Myron. "The valuation of Option Contracts and a Test of Market Efficiency." *J. Finance* 27. May 1972: 399-417.
- Black, Fisher. "How we came up with the Option formula". *Journal of Portfolio Management*. Winter 1989:23-36
- Black, Fisher. "The pricing of options and corporate Liability", *J. of Political Economy*, 1972: 636-653
- Breen, Richard. "The Accelerated Binominal Option Pricing model". *JFQ*. June 1991: 153-64
- Bunch, David S. Johnsen, Herb "A simple and Numerically efficient valuation method for American puts using a modified Geske-Johnsen approach". *J. of Finance*:809-16
- Flood, Mark D. "On the use of option pricing models to analyze deposits Insurance". *Federal Reserve Bank of St. Louis*. Jan/Feb 1990: 19-35
- Fama, E. F. "Dividend policy". *J. of Am. Statistics*. September 1968: 1132-62
- Figlewski Stephen. "What does an Option Pricing Model tell us about Option prices?". *Financial Analysts Journal*. Sept/Oct 1989: 12-15
- Figlewski, Silber, S, Subrahmanyam, M. "Financial Options: from theory to practice", Irwin, 1990
- Gastineau, Gary L.. "The Option Manual". Mc-Graw Hill, 1988
- Hull, John. "Option, Futures and other derivative securities". Prentice Hall, 1993
- Kamrad, Bardia. "Multinomial approximating Models for Options with k state variables". *Management Science*. Dec 1991: 1640-52.
- Kremer, Joseph. "Warrent pricing: Jump-diffusion vs. Black-Scholes". *J. of Financial & Quantitative Analysis*. June 1993: 255-72
- Murphy, J. Austin. "A modification and re-examination of the Bachelier Option Pricing Model". *American Economist*. Fall 1990: 34-41
- Nusse, H.E. "Resolution of chaos with application to a modified samuelson Model". *Journal of Economic Dynamics 1 Control*. Feb 1990:1-19
- Pindycke and Rubinfeld. "Econometric Models and economic forecasts". Mc-Graw Hill, 1991
- Porter, Michael, Miller, Thomas R. "Uses and abuses of financial modeling". *Topics in health care Financing*. Fall 1992: 34-45
- Samuelson, Douglas A. "On the hazards of relying on mathematical Models: Three cases from Federal Policy Analysis". *Computers & Operations Research*. 1991: 603-6
- Schroder, Mark "A reduction method applicable to compound option formulas". *Management Science*. Jul 1989: 823-27
- Tian, Yisong. "A modified lattice approach to option pricing". *Journal of futures Markets*. Aug 1993:563-77

SECONDARY REFERENCES

- Aliprantis, Charalambos D.; Brown, Donald J. Et al. "valuation and optimality in the overlapping generations model. *International economic review*. 1990; 31: 275-288
- Batina, Raymond G. "On the interpretation of the modified Samuelson rule for public goods in static models with heterogeneity". *Journal of Public Economics*. 1990: 42: 125-133
- Bergstarnd, jeffrey H. "The Heckscher-Ohlin-Samuelson Model, the Linder Hypothesis and the determinants of Bilateral Intra-Industry Trade". *Economiv Journal*. 1990: 100: 1216-1229
- Flood, Mark D. "On the use of option pricing models to analyze deposit insurance". *Federal reserve Bank of St. Louis review*. 1990: 72: 19-35
- Landon, Stuart. "The two-sector model and production technique in the short run and long run. *Oxford economic papers*. 1990: 42: 429-445
- Maloney, Kevin J., Byrne, Mark J. "An equilibrium debt option pricing model in discrete time". *Journal of banking & Finance*. 1989: 13: 421-442
- Myers, Stewart C. "Merton H. Miller's contributions to financial economics: bibliography of Merton H. Miller's publications. 1991: 93: 23-36
- Wolf, Avner. "How to use pricing models for foreign exchange options. *Futures: the magazine of commodities & options*. 1989; 18: 48, 50
- Quigg, Laura. "Empirical testing of real option-pricing models". *Journal of finance*. 1992: 48: 621-640